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FEEBLY NIL-CLEAN UNITAL RINGS

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ABSTRACT. We define and explore *feebly nil-clean rings* in different aspects. Our results considerably enlarge in certain aspects recent theorems published in J. Algebra and its Applications by various authors quoted in the bibliography list. Specifically, we prove that any feebly nil-clean ring R is decomposed as the direct product of a nil-clean ring and a 3-good ring in which 3 is a nilpotent. In some partial cases, such a ring R can be completely characterized. We also show that the full matrix $n \times n$ ring $\mathbb{M}_n(F)$ over a field F is feebly nil-clean if, and only if, either $F \cong \mathbb{Z}_2$ or $F \cong \mathbb{Z}_3$.

1. INTRODUCTION AND BACKGROUND

Everywhere in the text, all our rings are assumed to be associative, containing the identity element 1 which differs from the zero element 0. Our terminology and notations are at most standard and mainly in agreement with those from [11]. For instance, $U(R)$ stands for the group of units in R, $J(R)$ for the Jacobson radical of R, $Nil(R)$ for the set of nilpotents in R and $Id(R)$ for the set of idempotents in R.

Before presenting our chief achievements, we will foremost give a brief history of the principally known things in the subject: In [12] were stated the famous *clean* rings as rings in which each element is the sum of a unit and an idempotent. Later on, in [9] were defined the so-called *nil-clean* rings as those rings in which every element is the sum of a nilpotent and an idempotent; these rings form a more restricted class. Further, in [4] nil-clean rings were naturally extended to *weakly clean* rings which are rings for which any element is the sum or the difference of a nilpotent and an idempotent. The transversal between these two classes of rings were discovered independently in [14] and [6] (cf. [7] too).

After that, there is an abundance of subsequent generalizations of weakly nil-clean rings as these in [1], [5] and [15]. In fact, in the firstly cited two articles were defined the so-called 2-*nil-clean* rings as those rings whose elements are sums of a nilpotent and two idempotents; if these three elements commute between

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them, then 2-nil-clean rings are said to be *strongly 2-nil-clean*. And finally, in the thirdly cited article were considered rings whose elements are sums of a nilpotent, idempotent and tripotent all commuting one to other.

On the other vein, in [2] was defined the concept of a feebly clean ring as follows: A ring R is said to be *feebly clean* if, for every $R \in R$, there exist $u \in U(R)$ and $e, f \in Id(R)$ with $ef = fe$ such that $r = u + e - f$. It is self-evident that clean rings are feebly clean, whereas this implication is not always reversible.

So, we are in a position to introduce our main notion which is a common generalization of weakly nil-clean rings.

Definition 1.1. We will say that a ring R is *feebly nil-clean* if, for each $r \in R$, there are $q \in Nil(R)$ and $e, f \in Id(R)$ with $ef = fe$ such that $r = q + e - f$.

In particular, if always $qe = eq$ (or, respectively, $qf = fq$), the feebly nil-clean ring R is called *strongly feebly nil-clean*. If, however, q commutes with both e and f, the strongly feebly nil-clean rings will be called *super strongly feebly nil-clean*.

Actually, we can assume even that $ef = fe = 0$ since $e-f = e(1-f)-f(1-e)$ and $e(1-f), f(1-e) \in Id(R)$ are obviously orthogonal. Moreover, Definition 1.1 can be equivalently restated thus: A ring R is feebly nil-clean exactly when, for each $r \in R$, there are $t \in Nil(R)$ and $g, h \in Id(R)$ with $r = t+g+h$ and $gh = hg$. This follows easily by tricking with the elements $r - 1$ and $r + 1$, respectively.

Moreover, one observes that in Definition 1.1 the element $e - f$ is always a tripotent, that is, $(e - f)^3 = e - f$. In that aspect, if we intend to generalize slightly this replacing $e - f$ by any tripotent t (an element with $t^3 = t$), we however should take in mind that t is the sum of two commuting idempotents whenever $3 = 0$, namely $t = (t - t^2) + t^2$. In fact, $t^2 = (t^2)^2$ as well as $(t - t^2)^2 = t^2 - 2t^3 + t^4 = 2t^2 - 2t = t - t^2$, as needed. That is why, in the case of characteristic 3, the new attempt of non-trivial generalization gives nothing new.

Likewise, it is worthwhile noticing that the next relationships are trivially fulfilled:

$$\{\text{strongly 2-nil-clean}\} \subseteq \{\text{strongly feebly nil-clean}\} \subseteq \{\text{feebly nil-clean}\}$$
$$\subseteq \{\text{feebly clean}\} \cap \{\text{2-nil-clean}\}.$$

However, these containments are strict and cannot be reversed. Indeed, by a direct computation it could be verified that an example of a feebly nil-clean ring which is not strongly 2-nil-clean is the matrix ring $\mathbb{M}_2(\mathbb{Z}_3)$. Even mush more, it is established in Theorem 2.13 below that for any $n > 1$ the full matrix $n \times n$ ring $\mathbb{M}_n(F)$ over a field F is feebly nil-clean if, and only if, either $F \cong \mathbb{Z}_2$ or $F \cong \mathbb{Z}_3$ thus somewhat strengthening [4, Theorem 20]. Also, simple calculations show that the triangular ring $\mathbb{T}_2(\mathbb{Z}_3)$ is indecomposable strongly 2-nil-clean (and thus indecomposable strongly feebly clean) which is not weakly nil-clean. Nevertheless, if a feebly clean ring possesses only trivial idempotents, it is weakly nil-clean itself. One also observes that feebly nil-clean rings of characteristic 2 are nil-clean always.

The motivation to write the present paper is to continue the investigations in [4], [1], [5] and [15] in the general **noncommutative** case. This is successfully done by a number of results – see, for instance, Proposition 2.4 and Theorem 2.12 proved in the sequel.

2. Main Results

We are now ready to proceed by proving our results, starting by a series of crucial technicalities.

Lemma 2.1. *A homomorphic image of a feebly nil-clean ring is again a feebly nil-clean ring.*

Proof. It is straightforward, and so we leave all details to the reader. □

Lemma 2.2. *Finite direct products of (strongly) feebly nil-clean rings are again (strongly) feebly nil-clean rings.*

Proof. It is rather elementary, and so we omit details. □

Lemma 2.3. *The Jacobson radical of every feebly nil-clean ring is nil.*

Proof. For such a ring R, let first we assume that $6 = 0$. Since by the first part of Proposition 2.4 below, $R \cong R_1 \times R_2$, where $char(R_1) = 2$ and $char(R_2) = 3$, we have $J(R) \cong J(R_1) \times J(R_2)$. But R_1 being nil-clean implies with the aid of [9] that $J(R_1)$ is nil. We now intend to show that the same is $J(R_2)$. To that aim, given z in $J(R_2)$, we write that $1 + z = q + e - f$, where $q \in Nil(R_2)$ and $e, f \in Id(R_2)$ with $ef = fe = 0$. Since $u := z + (1 - q) \in U(R_2)$ and $(e - f)^3 = e - f$, it follows that $u = e - f$ whence $u^3 = u$ and so $u^2 = 1$. Therefore, $(z + (1 - q))^2 = ((z - q) + 1)^2 = 1$, i.e., $(z - q)^2 + 2(z - q) = 0$, i.e., $(z - q)^2 = z - q$. Thus $z - q = b$ for some idempotent b. Consequently, $1 - b = (1 + q) - z \in U(R_2) \cap Id(R_2) = \{1\}$. Now, $b = 0$ and hence $z = q$ is nilpotent, as claimed.

To treat the general case, since in view of the first half of Proposition 2.4 below 6 is a central nilpotent, $6R$ is contained in $J(R)$. Moreover, it is easily checked that $J(R)/6R$ is contained in $J(R/6R)$, hence by what we have just shown above it is nil. Now, it is easy to see that every element of $J(R)$ is nilpotent, as desired. □

Recall that a ring is said to be 3-*good* if every its element is the sum of three units. We now have accumulated all the information necessary to establish the following.

Proposition 2.4. *In any feebly nil-clean ring R the element 6 is nilpotent and so R is either a nil-clean ring or is a 3-good ring in which 3 is nilpotent or decomposes into the direct product of two such rings.*

Proof. Writing $2-q=e-f$, we thus have $(2-q)^2=e+f$ and hence $6+t=2e$ for some nilpotent t. Furthermore, $(6+t)^2=4e=2(6+t)$ whence 24 is nilpotent, which implies that $6^3=24.9$ is nilpotent, too. Finally, 6 must be a nilpotent, as claimed.

That is why, using well-known arguments, we may write $R=R_1\times R_2$, where by Lemma 2.1 first factor R_1 is either zero or a feebly clean ring in which 2 is a nilpotent (and so $2\in J(R_1)$), and the second factor R_2 is either zero or a feebly clean ring in which 3 is a nilpotent (and so $3\in J(R_2)$). Furthermore, again in virtue of Lemma 2.1, the quotient $R_1/J(R_1)$ is feebly nil-clean of characteristic 2 which as commented above enables us that $R_1/J(R_1)$ is nil-clean. But Lemma 2.3 allows us to deduce that R_1 is nil and consequently [9] applies to get that R_1 is nil-clean, indeed.

As for the latter quotient $R_2/J(R_2)$, we assert that it is 3-good being of characteristic 3. To that goal, it is enough to demonstrate that any feebly nil-clean ring of characteristic 3 is 3-good. In fact, writing $r=q+e-f$ in the usual feebly nil-clean decomposition, it can be written that $r=q+(e+f)-2f=(q+1)+(1+e+f)+(1-2f)$ as a sum of three units (to be precise as the sum of a unipotent and two involutions), because $e+f$ is an idempotent so that $1+(e+f)$ is an involution.

If now $x\in J(R_2)$ and $3\in J(R_2)$, it follows that $2\in U(R_2)$ and so we may write $x=(x-2)+1+1$ as the sum of three units. We then can infer that R_2 is 3-good as well, because units always lift modulo the Jacobson radical. □

It follows from the results above that if R is a feebly nil-clean ring, then $J(R)$ is nil and $R/J(R)$ is feebly nil-clean (as well as $6R$ is a nil-ideal and $R/6R$ is feebly nil-clean). Of some interest and importance is whether or not the converse implications are also valid. In some cases this is true – for example, let $T=\mathbb{T}_2(\mathbb{Z}_3)$. It is principally well known that T is an indecomposable feebly nil-clean ring with index of nilpotence 2, $J(T)$ is nil and $T/J(T)\cong\mathbb{Z}_3\times\mathbb{Z}_3$ is a decomposable feebly nil-clean ring.

This suggests us to come to the following reduction equivalence.

Theorem 2.5. *Suppose R is a ring with either $2 \in J(R)$ or $3 \in J(R)$. Then R is feebly nil-clean if, and only if, $J(R)$ is nil and $R/J(R)$ is feebly nil-clean.*

Proof. Since the necessity follows directly from a simple combination of Lemmas 2.3 and 2.1, we shall restrict our attention on the sufficiency. To that purpose, assume first that $2 \in J(R)$. As noted above, $R/J(R)$ is necessarily nil-clean and the claim follows directly from [9].

Next, assume $3 \in J(R)$. Take $r \in R$. If $r + 1 \in J(R)$, then there is nothing to prove. But if $r + 1 \in R \setminus J(R)$, then $r + J(R) \in R/J(R)$ and so we may write $r+1+J(R) = (a+J(R))+(x+J(R))-(y+J(R))$, where $a+J(R) \in Nil(R/J(R))$ with $a \in R$ and $x + J(R), y + J(R) \in Id(R/J(R))$ with $(x + J(R))(y + J(R)) = (y + J(R))(x + J(R)) = J(R)$; $x, y \in R$. Since $J(R)$ is nil, it easily follows that $a \in Nil(R)$. Moreover, both $xy, yx \in J(R)$. Denote $x - y = d$. Clearly $d^3 \in d+J(R)$. Furthermore, one computes that $(d+2d^2)^2 = d^2+4d^3+4d^4 \in d^2+d^3+d^4+J(R) = d+2d^2+J(R)$, whence it is a well-known folklore fact that there is an idempotent $e \in R$, which is a polynomial of $d+2d^2$ and hence of d, with the property $e \in d+2d^2+J(R)$. Similarly, $(-2d^2)^2 = 4d^4 \in 4d^2+J(R) = -2d^2+J(R)$ and so there is an idempotent $f \in R$, which is a polynomial of $-2d^2$ and hence of d, with the property $f \in -2d^2 + J(R)$. Finally, $ef = fe$ and $d = e + f$. But $Nil(R) + J(R) = Nil(R)$ whenever $J(R) \subseteq Nil(R)$, so that now we can write $r = b + e - (1 - f)$ for some $b \in Nil(R)$. However, $r = b + ef - (1 - f)(1 - e)$ with $ef, (1 - f)(1 - e) \in Id(R)$ being orthogonal idempotents, as required. □

A natural query which immediately arises is whether or not the preceding statement remains true in all generality, i.e., when $6 \in J(R)$ (compare with Proposition 2.4). We now will prove this in some partial cases.

Proposition 2.6. *Let R be a ring whose $R/J(R)$ is indecomposable ring. Then R is feebly nil-clean if, and only if $J(R)$ is nil and $R/J(R)$ is feebly nil-clean.*

Proof. As above, we will consider only the sufficiency. Thus, in virtue of Proposition 2.4, either $2 \in J(R)$ or $3 \in J(R)$. We hereafter may employ Theorem 2.5 to get the desired claim. □

Proposition 2.7. *Any strongly feebly nil-clean ring is clean.*

Proof. Write $r = q + e - f$ as a feebly nil-clean decomposition of r. If $qe = eq$, it follows that $r = [q + (2e - 1)] + [1 - (e + f)]$ is a clean decomposition for r. Reciprocally, if $qf = fq$, then one may presenting $r = [q+(1-2f)]-[1-(e+f)]$ as a clean element. □

Remark 2.8. We shall say that a ring R is *ω-clean* if, $\forall\ r \in R$, $\exists\ u \in U(R)$ with $u^n = 1$ for some $n \in \mathbb{N}$ and $\exists\ e \in Id(R)$: $r = u + e$.

One sees now that any strongly feebly nil-clean ring of characteristic 3 is ω-clean. In fact, there is $n \in \mathbb{N}$ with $q^{3^n} = 0$ and, therefore, it is not too hard to verify that $[q + (2e-1)]^{2.3^n} = 1$, as wanted.

We can process by a way of similarity for the element $q + (1-2f)$, thus concluding our claim.

Imitating [13], a ring R is said to be an *NS-ring* if $Nil(R)$ is a subring of R. It follows that a ring R is NS if, and only if, $1 + Nil(R)$ is a subgroup of $U(R)$. Thus all UU rings (that are rings R for which $U(R) = 1 + Nil(R)$) are always NS rings. It was proven in [13] that exchange NS rings are reduced modulo their Jacobson radical. We shall further use this in order to prove the following:

Proposition 2.9. *A ring R is strongly feebly nil-clean and NS if, and only if, $J(R) = Nil(R)$ and $R/J(R)$ is embedding in the direct product of copies of $\mathbb{Z}_2$ and $\mathbb{Z}_3$.*

Proof. "**Necessity**". With Proposition 2.7 at hand, accomplished with the aforementioned fact from [13], we deduce that the quotient $R/J(R)$ must be reduced, that is, $Nil(R) \subseteq J(R)$. But this means that $J(R) = Nil(R)$ because by Lemma 2.3 we have that $J(R)$ is nil. Furthermore, every element in $R/J(R)$ is the difference of two commuting idempotents which obviously amounts to the fact that each element of $R/J(R)$ is the sum of two idempotents. We consequently apply [10, Theorem 1] to conclude the pursued claim.

"**Sufficiency**". Since $1 + Nil(R) = 1 + J(R)$ is always a subgroup of $U(R)$, we derive that R is an NS-ring. We shall show now that R is even strongly 2-nil-clean and thus, as observed above, it will be strongly feebly nil-clean as well. In fact, we appeal to [5, Theorem 3.3] (see [8] for more account too). □

Remark 2.10. If only the second direct factor R_2, in the decomposition of the feebly nil-clean ring $R \cong R_1 \times R_2$, where R_1 is nil-clean, would be an NS-ring, then $R_2/J(R_2) \cong P \subseteq \prod_\lambda \mathbb{Z}_3$.

It is well known that the center of a clean ring need not be clean. The following surprising affirmation somewhat extends [4, Proposition 10].

Proposition 2.11. *The center of a feebly nil-clean ring in which $6 = 0$ is feebly nil-clean as well.*

Proof. Suppose R is a feebly nil-clean ring with $6 = 0$. Invoking Proposition 2.4, $R \cong R_1 \times R_2$, where either $R_1 = \{0\}$ or R_1 is nil-clean of characteristic 2, and either $R_2 = \{0\}$ or R_2 is feebly nil-clean of characteristic 3. Since $Z(R) = Z(R_1) \times Z(R_2)$, and by [4, Corollary 11] the ring $Z(R_1)$ is nil-clean, it suffices to show that $Z(R_2)$ is feebly nil-clean. Henceforth, Lemma 2.2 will apply to get the wanted assertion.

So, given $z \in Z(R_2)$, we write $z = c + g - h$, where $g, h \in Id(R_2)$ with $gh = hg = 0$ and $c \in Nil(R_2)$ choosing such an $n \in \mathbb{N}$ that $c^{3^n} = 0$. Writing $z - c = g - h$, it follows that $z^{3^n} = (z - c)^{3^n} = (g - h)^{3^n} = g - h = z - c$. Thus $c = z - z^{3^n} \in Z(R_2)$. This implies, moreover, that $g - h \in Z(R_2)$ and by squaring we deduce that $g + h \in Z(R_2)$. That is why, summarizing both equalities, we infer that $-g = 2g \in Z(R_2)$ yielding that $g \in Z(R_2)$. Finally, one sees that $h \in Z(R_2)$, as required. □

A new question which immediately arises is whether or not the last statement could be extended in the general case (remember it must be here that $6 \in J(R)$).

We now come to our central characterization criterion for a proper subclass of the class of feebly nil-clean rings.

Theorem 2.12. *A ring R is super strongly feebly nil-clean if, and only if, $J(R)$ is nil and $R/J(R)$ is embedding in the direct product of copies of $\mathbb{Z}_2$ and $\mathbb{Z}_3$.*

Proof. We shall show that super strongly feebly nil-clean rings do actually coincide with strongly 2-nil-clean rings, and so [5, Theorem 3.3] (cf. [8] as well) works to get the assertion. In fact, this follows via the next trick: If R is strongly 2-nil-clean, then for any $r \in R$ we may write $r+1 = q+e+f$, where q is nilpotent, e, f are idempotents and all of them commute. But now $r = q + e - (1 - f)$, where $1 - f$ is obviously an idempotent which commutes with e, as required. Conversely, if R is super strongly feebly nil-clean, then for any $r \in R$ we have $r - 1 = q + e - f$, where q is nilpotent, e, f are idempotents and all of them commute. Now $r = q + e + (1 - f)$, where again $1 - f$ is an idempotent which commutes with e, as needed. □

We finish off our basic statements with the following one:

Theorem 2.13. *Let $n \geq 2$. Then the ring $\mathbb{M}_n(F)$ over a field F is feebly nil-clean if, and only if, either $F \cong \mathbb{Z}_2$ or $F \cong \mathbb{Z}_3$.*

Proof. "$\Rightarrow$". Letting $a \in F$ be an arbitrary element, then we write $aI_n = Q + E_1 - E_2$, where I_n is the identity matrix, Q is a nilpotent matrix and E_1, E_2 are idempotent matrices with $E_1.E_2 = E_2.E_1 = 0_n$, where 0_n is the zero matrix. Since aI_n is the scalar matrix which is a central matrix element, it is obvious that Q commutes with both E_1 and E_2. But the element a being invertible yields that

aI_n is an invertible matrix, and hence so does $aI_n - Q = E_1 - E_2$. Therefore, $(E_1+E_2)^2 = E_1+E_2 = I_n$, because $(E_1-E_2)^3 = E_1-E_2$ whence $(E_1-E_2)^2 = E_1+E_2 = I_n$. However, $aI_n = Q + I_n - 2E_2$, i.e., $(a-1)I_n - Q = -2E_2$. By squaring and comparing, we conclude that $(a-1)^2I_n + 2(a-1)I_n = (a^2-1)I_n$ must be a nilpotent. Thus $a^2-1 \in Nil(F) = \{0\}$. This means that $(a-1)(a+1) = 0$, that is, $a = 1$ or $a = -1$. If $-1 = 1$, then $F = \{0,1\}$. In the other case when $1 \neq -1$, we have $F = \{0,1,-1\}$, as expected.

"$\Leftarrow$". For $F \cong \mathbb{Z}_2$ it was proved in [3] that the ring $\mathbb{M}_n(\mathbb{Z}_2)$ is nil-clean and hence feebly nil-clean.

Suppose now that $F \cong \mathbb{Z}_3$. We will prove that for every matrix A we can find a decomposition $A = N - E_1 - E_2$ or $A = N + E_1 + E_2$ such that N is nilpotent and E_1, E_2 are idempotent matrices with $E_1E_2 = E_2E_1$; thus replacing A by $A - I_n$ in the first case or A by $A + I_n$ in the second one, we shall obtain that $A = N + (I_n - E_1) - E_2$ or that $A = N + E_2 - (I_n - E_1)$, as required in Definition 1.1. Since this property is preserved by the similarity relation, we can replace A by its Frobenious normal form $\mathrm{diag}(C_1, \dots, C_k)$, where C_i are companion matrices. Therefore, it is easy to see that it is enough to prove that every companion matrix

$$\text{(C)} \qquad C = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -c_{m-2} \\ 0 & 0 & \cdots & 1 & -c_{m-1} \end{pmatrix} \in \mathbb{M}_n(\mathbb{Z}_3)$$

has the desired decomposition.

Case I: If $c_{m-1} = 1$, then we take

$$E_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & c_0 \\ 0 & 0 & \cdots & 0 & c_1 \\ 0 & 0 & \cdots & 0 & c_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & c_{m-2} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, E_2 = 0 \text{ and } N = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Case II: If $c_{m-1} = -1$, then we take

$$E_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 0 & 0 & \cdots & 0 & -c_1 \\ 0 & 0 & \cdots & 0 & -c_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -c_{m-2} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = E_2 \text{ and } N = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Case III: If $c_{m-1} = 0$ and $m \geq 4$, then we take

$$E_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 2 & 2 \\ 0 & 0 & \cdots & 2 & 2 \end{pmatrix} \text{ and } E_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

It is easy to verify that in each of these cases $N = C + E_1 + E_2$ is a nilpotent and $E_1E_2 = E_2E_1$, as wanted.

If now $m = 2$, then $c_1 = 0$ and $C = \begin{pmatrix} 0 & -c_0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -c_0 - 1 \\ 0 & 0 \end{pmatrix} = E_1 + E_2 + N$, where the first two matrices are commuting idempotents, whereas the third one is obviously a nilpotent.

Next, if $m = 3$, then $c_2 = 0$ and $C = \begin{pmatrix} 0 & 0 & -c_0 \\ 1 & 0 & -c_1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -c_0 \\ 0 & 0 & -c_1 - 1 \\ 0 & 0 & 0 \end{pmatrix} = E_1 + E_2 + N$, where the first two matrices are commuting idempotents as surprisingly $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$, while the third one is obviously a nilpotent. □

3. Open Questions

We close the work with a few problems of interest:

Problem 3.1. Is it true that a (strongly) feebly nil-clean of characteristic 3 is a subdirect product of copies of the field $\mathbb{Z}_3$?

Problem 3.2. Does it follow that feebly nil-clean rings are clean or exchange?

Notice that by Proposition 2.7 strongly feebly nil-clean rings are themselves clean.

Mimicking [6], recall that a ring R is said to be *WUU* if $U(R) = Nil(R) \pm 1$.

Problem 3.3. Is a ring R feebly nil-clean and WUU if, and only if, R is super strongly feebly nil-clean?

Problem 3.4. Describe *uniquely feebly nil-clean rings* in the sense of feebly nil-clean rings for which the existing idempotents are unique (that is, the existing feebly nil-clean presentation is unique). Are they abelian feebly nil-clean?

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